

ON HARMONIC ELEMENTS FOR SEMI-SIMPLE LIE ALGEBRAS

Philippe CALDERO

ABSTRACT. *Let \mathfrak{g} be a semi-simple complex Lie algebra, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ its triangular decomposition. Let $U(\mathfrak{g})$, resp. $U_q(\mathfrak{g})$, be its enveloping algebra, resp. its quantized enveloping algebra. This article gives a quantum approach to the combinatorics of (classical) harmonic elements and Kostant's generalized exponents for \mathfrak{g} . A quantum analogue of the space of harmonic elements has been given in [21]. On the one hand, we give specialization results concerning harmonic elements, central elements of $U_q(\mathfrak{g})$, and the Joseph and Letzter's decomposition, see [21]. For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, we describe the specialization of quantum harmonic space in the \mathbb{N} -filtered algebra $U(\mathfrak{sl}_{n+1})$ as the materialization of a theorem of Lascoux-Leclerc-Thibon, [29]. This enables us to study a Joseph-Letzter decomposition in the algebra $U(\mathfrak{sl}_{n+1})$. On the other hand, we prove that highest weight harmonic elements can be calculated in terms of the dual of Lusztig's canonical base. In the simply laced case, we parametrize a base of \mathfrak{n} -invariants of minimal primitive quotients by the set \mathcal{C}_0 of integral points of a convex cone.*

This research has been partially supported by the EC TMR network "Algebraic Lie Representations" , contract no. ERB FMTX-CT97-0100.

0. Introduction.

0.1. Let V be a complex finite dimensional space and G a Lie subgroup of $GL(V)$. Assume that V is completely reducible as a G -module. Then G acts semi-simply on the symmetric \mathbb{C} -algebra $S(V)$. Let J be the ideal generated by the non-constant homogeneous G -invariant elements of $S(V)$. Let V^* be the dual of V . There exists a natural non-degenerate duality between $S(V)$ and $S(V^*)$. Let $H^* \subset S(V^*)$ be the orthogonal of J , H^* is the space of G -harmonic polynomials of V .

Suppose now that G is a complex simply connected semi-simple Lie group acting by the adjoint action on its associated Lie algebra $\mathfrak{g} = V$. Then, the Killing form maps the harmonic space H^* onto $H \subset S(\mathfrak{g})$. We have $S(\mathfrak{g}) = H \oplus J$, cf [27].

Let $U(\mathfrak{g})$ be the enveloping algebra of \mathfrak{g} and $Z(\mathfrak{g})$ be its center. Recall that $U(\mathfrak{g})$ is naturally filtered and the associated graded algebra of $U(\mathfrak{g})$ is $S(\mathfrak{g})$, by the Poincaré-Birkhoff-Witt theorem. As in [27], we can choose a sub- $\text{ad}\mathfrak{g}$ -module \mathcal{H} of $U(\mathfrak{g})$ such the graded space associated to \mathcal{H} is H . The space \mathcal{H} will be called harmonic space in $U(\mathfrak{g})$. Kostant's separation theorem states that $U(\mathfrak{g}) \simeq Z(\mathfrak{g}) \otimes \mathcal{H}$ via multiplication.

0.2. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition of \mathfrak{g} . Let P be the group of integral weights of \mathfrak{g} and P^+ be the semi-group of dominant weights of \mathfrak{g} , generated by the fundamental weights ϖ_i , $1 \leq i \leq n$. The set Δ , resp. Δ^+ , is the root system, resp. the set of positive roots, and W is the Weyl group. For all λ in P^+ , let $V(\lambda)$ be the simple \mathfrak{g} -module with highest weight λ .

Let's sketch a few results on H that can be found in [27]. First of all, H is a $\text{ad}\mathfrak{g}$ -module and the multiplicity of $V(\lambda)$ in H is

$$(0.2.1) \quad k_\lambda := [H : V(\lambda)] = \dim V(\lambda)_0,$$

where $V(\lambda)_0$ is the 0-weight subspace of $V(\lambda)$. Let $H(\lambda)$ be the isotypical component of $V(\lambda)$ in H and let $S(\mathfrak{g})^n$ be the n -th graded component of $S(\mathfrak{g})$. Then, there are integers $m_\lambda^1 \leq \dots m_\lambda^{k_\lambda}$ such that

$$(0.2.2) \quad \sum_{n \geq 0} [H(\lambda) \cap S(\mathfrak{g})^n : V(\lambda)] t^n = \sum_{i=1}^{k_\lambda} t^{m_\lambda^i},$$

as polynomials in t . The m_λ^i are called Kostant's generalized exponents. Indeed, they generalize the usual exponents of \mathfrak{g} because they can be realized as exponents of the eigenvalue of the Coxeter element acting on the zero weight space of $V(\lambda)$. Two problems arise :

Problem 1. Find some combinatorial rules that calculates the generalized exponents.

Problem 2. Calculate the harmonic elements of $U(\mathfrak{g})$.

0.3. Let's sketch results about Problem 1. For $\mu \in P$, let $\mathcal{P}(\mu)$ be the dimension of the space of weight μ in $S(\mathfrak{n})$. \mathcal{P} is Kostant's partition function. In the completed group algebra $\mathbb{C} \langle P \rangle$, cf. [14, 7.5], we have $\sum_{\mu \in P} \mathcal{P}(\mu) e^{-\mu} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1}$. Lusztig's q -analogue of Kostant's partition function is given by $\sum_{\mu \in P} \mathcal{P}_q(\mu) e^{-\mu} = \prod_{\alpha \in \Delta^+} (1 - qe^{-\alpha})^{-1}$, where q is an indeterminate. For $\lambda \in P^+$ and $\mu \in P$, the Kostka-Foulkes polynomial is defined by

$$(0.3.1) \quad K_{\lambda\mu}(q) = \sum_{w \in W} (-1)^{l(w)} \mathcal{P}_q(w(\lambda + \rho) - \mu - \rho),$$

where $l(w)$ is the length of w and ρ is the sum of the fundamental weights. By [14, 7.5.10], $K_{\lambda\mu}(1) = \dim V(\lambda)_\mu$. R. Brylinski gives the following interpretation of Hesselink formula :

Theorem. [17], [7] For $\lambda \in P^+$, $K_{\lambda 0}(q) = \sum_{i=1}^{k_\lambda} q^{m_\lambda^i}$.

But this is not an answer to Problem 1 because of the signs in (0.3.1).

In [29], there is given some nice combinatorics for the generalized exponents in the case A_n : first a multi-variable Kostka-Foulkes function $\mathbb{K}_{\lambda 0}(X_1, \dots, X_n)$ is defined in terms of the set $B(\lambda)_0$ of zero weight elements in the crystal base of $V(\lambda)$ by

$$\mathbb{K}_{\lambda 0}(X_1, \dots, X_n) = \sum_{b \in B(\lambda)_0} \prod_{i=1}^n X_i^{\varepsilon_i(b)},$$

where ε_i are the usual parameters of the crystal base. It can be easily calculated by the combinatorics described in [25]. Then, the Kostka-Foulkes function $K_{\lambda 0}(q)$ is recovered by specializing X_i on q^i :

Theorem. [29] For $\lambda \in P^+$, $K_{\lambda 0}(q) = \mathbb{K}_{\lambda 0}(q, q^2, \dots, q^n)$.

So much for Problem 1 in the A_n case. Now, Problem 2 has no known general solution even in \mathfrak{sl}_3 .

0.4. The article aims to present a quantum theory of harmonic spaces for enveloping algebras of semi-simple Lie algebras, in order to use it to get results in the classical case.

Let q be an indeterminate. Let $U_q(\mathfrak{g})$ be the q -deformation of the enveloping algebra $U(\mathfrak{g})$. For λ in P^+ , let $V_q(\lambda)$ be the quantum simple module with highest weight λ .

The algebra $U_q(\mathfrak{g})$ contains the algebra $F(U_q(\mathfrak{g}))$ of ad-finite elements of $U_q(\mathfrak{g})$, see 1.4. We use Joseph-Letzter's decomposition theorem as a basic tool, [21]. It asserts that

$$(0.4.1) \quad F(U_q(\mathfrak{g})) = \bigoplus_{\lambda \in P^+} F_q^\lambda,$$

where F_q^λ are sub- $\text{ad}U_q(\mathfrak{g})$ -modules of $F(U_q(\mathfrak{g}))$ isomorphic to $V_q(\lambda)^* \otimes V_q(\lambda)$ endowed with the diagonal action. This can be seen, [9], as a Peter-Weyl theorem embedded in $U_q(\mathfrak{g})$ via the Rosso form, [34]. The center Z of $U_q(\mathfrak{g})$ is as in the classical case a polynomial algebra with dimension $n = rk(\mathfrak{g})$. The invariant elements z_i of $F_q^{\varpi_i}$, where $1 \leq i \leq n$, are algebraically independent generators of Z . Let J_q be the ideal in $F(U_q(\mathfrak{g}))$ generated by the z_i . By definition, an harmonic space H_q will be a complementary $\text{ad}U_q(\mathfrak{g})$ -module of J_q in $F(U_q(\mathfrak{g}))$, such that $H_q = \bigoplus_{\lambda \in P^+} F_q^\lambda \cap H_q$, with compatibility conditions, see [1.8, Definition 1].

0.5. The second tool is what we call Ringel's filtration : a Poincaré-Birkhoff-Witt base of $U_q(\mathfrak{g})$ can be lexicographically ordered in such a way that the graded algebra associated to this ordering, see 1.3, is q -commutative. We prove that, for this ordering :

Theorem 1. *The set $\{z_i, 1 \leq i \leq n\}$ is a Gröbner base of the ideal J_q .*

We obtain that $U_q(\mathfrak{g})$ is free over its center, with a set of explicit generators given in terms of roots packages, [11]. We prove an analogue of this result for quantized enveloping algebras at a root of one.

Theorem 2. When $q = \varepsilon$ is a root of one, then the algebra U_ε , see [12, 1.5], is free over its center.

As before, an explicit set of generators can be given.

0.6. For $\lambda \in P^+$, let $H_q(\lambda)$ be the isotypical component of type $V_q(\lambda)$ in H_q . A result of Joseph and Letzter asserts that (0.2.1) holds in the quantum case. Now, Joseph-Letzter's decomposition, see (0.4.1), affords a natural P^+ -filtration on $F(U_q(\mathfrak{g}))$. By analogy with equation (0.2.2), we define P^+ -exponents of λ as the family $\mu_\lambda^1, \dots, \mu_\lambda^{k_\lambda}$ of dominant weights such that

$$\sum_{\mu \in P^+} [H_q(\lambda) \cap \text{ad}U_q(\mathfrak{g}).K_{-2\mu} : V_q(\lambda)]e^\mu = \sum_{i=1}^{k_\lambda} e^{\mu_\lambda^i}$$

in the group algebra of P . We prove, see Proposition 1.8 and Proposition 3.4, that the quantum analogues of Problem 1 and Problem 2 can be solved in terms of Lusztig's canonical base and its dual :

Proposition. Let μ in P^+ . Let \mathcal{B}_H be the set of elements b of the canonical base such that $\varepsilon_i(b) \leq -(\text{wt}(b), \alpha_i^\vee)$, $1 \leq i \leq n$. Then,

- (i) $\sum_\lambda [H_q \cap F_q^\lambda : V_q(\mu)] \prod_i X_i^{<\lambda, \alpha_i^\vee>} = \mathbb{K}_{\mu 0}(X_1, \dots, X_n)$,
- (ii) a base of the space of \mathfrak{n} -invariants of H_q is labelled by \mathcal{B}_H .

Note that (i) can be also easily deduced from a result of Baumann, [1]. In particular, for $\mathfrak{g} = \mathfrak{sl}_{n+1}$, the multi-variable Kostka-Foulkes function $\mathbb{K}_{\lambda 0}(X_1, \dots, X_n)$ can be described as the Poincaré polynomial of $H_q(\lambda)$ in the P^+ -Joseph-Letzter filtration.

0.7. Another related problem is the following :

Problem 3. Describe a Joseph-Letzter decomposition inside the classical enveloping algebra $U(\mathfrak{g})$ endowed with its natural filtration.

This includes the problem of specialization of harmonic elements as well as the specialization of the center. For general \mathfrak{g} , we give some preparation theorems. Let A be the algebra $\mathbb{C}[q]$ localized at $q = 1$. We present a A -form H_A which links the quantum harmonic space with the classical one. Indeed, it verifies $H_q = \mathbb{C}(q) \otimes_A H_A$ and $H_1 = \mathbb{C} \otimes_A H_A$ which both verify the separation theorem at, respectively, the quantum and the classical level. We prove a separation theorem on A -forms.

For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, the understanding of the specialization of a) the quantum harmonic space and b) the center, see [8], enables us, by the separation theorem, to give an answer to Problem 3, see Theorem 4.3.

0.8. Recall that any reduced decomposition of the longest element $w_0 \in W$ in the Weyl group gives rise to a Poincaré-Birkhoff-Witt basis of $U_q(\mathfrak{n})$, whose elements are labelled by points in \mathbb{N}^N , where $N = \dim \mathfrak{n}$. When \mathfrak{g} has type A-D-E, then by [31], [32], the dual of the canonical base is in bijection with the PBW basis, with nice multiplicative properties, see (3.5.1)-(3.5.3). Hence, each element of the canonical base corresponds to an integral point in \mathbb{R}^N , where $N = \dim \mathfrak{n}$.

Theorem. Let $\mathcal{C}_0 \subset \mathbb{N}^N$ be the image of \mathcal{B}_H via the correspondence above. Then, \mathcal{C}_0 is the set of integral points in a convex cone.

We describe a degenerescence of the variety $\mathbb{C}[\mathcal{C}_0]$ which relates the combinatorics of Point 2, see Theorem 3.5.

1. Preliminaries and notations.

1.1. Let \mathfrak{g} be a semi-simple Lie \mathbb{C} -algebra of rank n . We fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition and $\{\alpha_i\}_i$ be a base of the root system Δ resulting from this decomposition. We note $\mathfrak{b} = \mathfrak{n} + \mathfrak{h}$ and $\mathfrak{b}^- = \mathfrak{n}^- + \mathfrak{h}$ the two opposite Borel sub-algebras. Let Δ^+ be the set of positive roots, P be the weight lattice generated by the fundamental weights ϖ_i , $1 \leq i \leq n$, and $P^+ := \sum_i \mathbb{N}\varpi_i$ the semigroup of integral dominant weights. P is endowed with the ordering \preceq

$$\lambda \preceq \mu \Leftrightarrow \mu - \lambda \in P^+.$$

We fix a *total* additive ordering \leq on P such that $\mu - \lambda \in Q^+ \Rightarrow \lambda \leq \mu$, $\lambda, \mu \in P$. Such an ordering always exists. For example, for $\lambda = \sum_i \lambda_i \alpha_i$, $\mu = \sum_i \mu_i \alpha_i$, $\lambda \leq \mu$ iff $(\sum_i \lambda_i, \lambda_1, \dots, \lambda_n)$ preceeds $(\sum_i \mu_i, \mu_1, \dots, \mu_n)$ for the lexicographical ordering of \mathbb{Q}^{n+1} verifies the hypothesis.

Let W be the Weyl group, generated by the reflections corresponding to the simple roots $s_i := s_{\alpha_i}$. Let w_0 be the longest element of W . We note $(,)$ a W -invariant form on P . For $\beta \in \Delta^+$, $\text{ht}(\beta)$ will mean the height of β .

1.2. Let q be an indeterminate and $U_q(\mathfrak{g})$ be the simply connected quantized enveloping algebra, defined as in [13, 0.2-0.3]. Let $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, be the subalgebra generated by the canonical generators E_{α_i} , resp. F_{α_i} , of positive, resp. negative, weights. For all λ in P , let K_λ be the corresponding element in the algebra $U_q^0 = \mathbb{C}(q)[P]$ of the torus of $U_q(\mathfrak{g})$. As in the classical case, we have the triangular decomposition : $U_q(\mathfrak{g}) \simeq U_q(\mathfrak{n}^-) \otimes U_q^0 \otimes U_q(\mathfrak{n})$.

$U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra with comultiplication Δ , antipode S and augmentation ε , [18, 3.2.9].

We define in $U_q(\mathfrak{g})$ the left adjoint action by $\text{ad } v.u = v_{(1)}uS(v_{(2)})$, where $\Delta(v) = v_{(1)} \otimes v_{(2)}$ with the Sweedler notations.

Let M be a $U_q(\mathfrak{g})$ -module and $\mu \in P$. Let M_μ be the space of elements of weight μ , i.e $M_\mu := \{u \in M, K_\alpha m = q^{(\alpha, \mu)} m, \forall \alpha \in P\}$.

1.3. We fix a decomposition of the longest element of the Weyl group $w_0 = s_{i_1} \dots s_{i_N}$, where $N = \dim \mathfrak{n}$. Set $\beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$, $1 \leq k \leq N$. We endow with an order the set Δ^+ of positive roots : $\beta_N < \dots \beta_2 < \beta_1 = \alpha_{i_1}$, see [12, 1.7]. For all β in Δ^+ , let E_β , resp. F_β , be the root elements of $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$. Set $E_i = E_{\alpha_i}$, $F_i = F_{\alpha_i}$, for each simple root. For each ψ in \mathbb{N}^N , set $E_\psi := \prod_{l=1}^N E_{\beta_l}^{\psi_l}$, resp. $F_\psi := \prod_{l=1}^N F_{\beta_l}^{\psi_l}$. For $\Gamma = (\phi, \lambda, \psi) \in \mathbb{N}^N \times P \times \mathbb{N}^N$, set $X_\Gamma := F_\phi K_{-\lambda} E_\psi$. We know, cf. [12, 1.6], that the E_ψ , resp. F_ϕ , resp. X_Γ , form a Poincaré-Birkhoff-Witt base of $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, resp. $U_q(\mathfrak{g})$, for $\psi \in \mathbb{N}^N$, resp. $\phi \in \mathbb{N}^N$, resp. $\Gamma \in \mathbb{N}^N \times P \times \mathbb{N}^N$. To $\Gamma = (\phi, \lambda, \psi) \in \mathbb{N}^N \times P \times \mathbb{N}^N$, we associate $\tilde{\Gamma} = (\sum(\phi_l + \psi_l)\text{ht}(\beta_l), \phi, \psi) \in \mathbb{N}^{2N+1}$. The set of subspaces

$$\left\{ \bigoplus_{\tilde{\Gamma}' \leq \tilde{\Gamma}} \mathbb{C}(q) X_{\Gamma'}, \Gamma \in \mathbb{N}^N \times P \times \mathbb{N}^N \right\},$$

for the lexicographic ordering on \mathbb{N}^{2N+1} define a filtration of algebra on $U_q(\mathfrak{g})$. In the sequel, this will be called the Ringel filtration, [33]. The associated graded $\text{Gr}U_q(\mathfrak{g})$ is generated by the $\text{Gr}F_\beta$, $\text{Gr}K_\lambda$, $\text{Gr}E_\beta$, where $\beta \in \Delta^+$, $\lambda \in P$, and q -commuting relations, [12, Proposition 1.7]. In particular, we have

$$(1.3.1) \quad \text{Gr}E_\alpha \text{Gr}E_\beta = q^{(\alpha, \beta)} \text{Gr}E_\beta \text{Gr}E_\alpha, \text{ if } \beta < \alpha$$

1.4. For all λ in P^+ , let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$ -module with highest weight λ and highest weight vector v_λ . Let $V_q(\lambda)^*$ be its dual, endowed with a structure of left $U_q(\mathfrak{g})$ -module twisted by the antipode. We define the ad-finite part of $U_q(\mathfrak{g})$ by

$$F(U_q(\mathfrak{g})) := \{u \in U_q(\mathfrak{g}), \dim \text{ad } U_q(\mathfrak{g})(u) < +\infty\}.$$

We have the theorem, [18, 7.1], [9] :

Theorem. $F(U_q(\mathfrak{g}))$ has the following properties :

- (i) $F(U_q(\mathfrak{g}))$ is an algebra and a semi-simple module for the adjoint action,
- (ii) $F(U_q(\mathfrak{g})) = \bigoplus_{\lambda \in P^+} \text{ad}U_q(\mathfrak{g}).K_{-2\lambda}$,
- (iii) The module $\text{ad}U_q(\mathfrak{g}).K_{-2\lambda}$ is isomorphic to $V_q(\lambda)^* \otimes V_q(\lambda)$. ◇

1.5. As in [11, 2.1], we define the root packages from a decomposition of w_0 .

Definition and notation. Fix $w_0 = s_{i_1} \dots s_{i_N}$. For $1 \leq j \leq n$, we call root packages the sets $\Delta_j^+ := \{\beta_l, i_l = j\}$. For m , $1 \leq m \leq k := \text{Card} \Delta_j^+$, we define $\alpha_{j,m}$ to be the m -th element in the decreasing sequence of the roots of Δ_j^+ : $\alpha_{j,1} > \alpha_{j,2} > \dots > \alpha_{j,m} > \dots > \alpha_{j,k}$.

Remark that for some decompositions of w_0 , the map $\alpha_{j,m} \mapsto \alpha_{j,m+1}$ corresponds to the translation functor in the Auslander-Reiten quiver, [3].

1.6. Let $(,)$ be the canonical pairing between $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{n})$, see [4, 1.2], \mathcal{B} be Lusztig's canonical base of $U_q(\mathfrak{n}^-)$, [loc. cit.], and $\mathcal{B}^* \subset U_q(\mathfrak{n})$ be the dual base, i.e. $(b^*, b') = \delta_{b,b'}$. Let $u \mapsto \bar{u}$ be the antihomomorphism of $U_q(\mathfrak{n}^-)$ such that $\bar{F}_i = F_i$ and $\bar{q} = q$. Let $\tilde{E}_i, \tilde{F}_i : U_q(\mathfrak{n}^-) \rightarrow U_q(\mathfrak{n}^-)$ be the Kashiwara operators, [loc. cit.]. For $b \in \mathcal{B}$, $\tilde{E}_i(b)$, resp. $\tilde{F}_i(b)$, equals some b' in $\mathcal{B} \cup \{0\}$ modulo $q^{-1}\mathbb{Z}[q^{-1}]\mathcal{B}$. The rule $b \mapsto b'$ defines maps \tilde{e}_i and \tilde{f}_i from \mathcal{B} to $\mathcal{B} \cup \{0\}$. For $b \in \mathcal{B}$, $1 \leq i \leq n$, set $\varepsilon_i(b) = \text{Max}\{r, \tilde{e}_i^r(b) \neq 0\}$, and $\mathcal{E}(b) = \sum_{i=1}^n \varepsilon_i(b) \varpi_i$. The following is well known, [24, Proposition 8.2], [18, 6.2.18-6.2.19] :

Theorem. For all $\lambda \in P^+$, we have :

- (i) Via the isomorphism of Theorem 1.4 (iii), $\text{ad}U_q(\mathfrak{n})K_{-2\lambda}$ is isomorphic to $V_q(\lambda)^* \otimes v_\lambda$ as a $U_q(\mathfrak{n})$ module,
- (ii) $\text{ad}U_q(\mathfrak{n})K_{-2\lambda}$ is generated as a space by $\{K_{-2\lambda}b^*, b^* \in \mathcal{B}(\lambda)^*\}$, where $\mathcal{B}(\lambda)^* := \{b^* \in \mathcal{B}^*, \mathcal{E}(\bar{b}) \leq \lambda\}$. ◇

1.7. Recall that the enveloping algebra $U := U(\mathfrak{g})$ is endowed with a canonical filtration $\{U_k, k \in \mathbb{N}\}$ such that the associated graded algebra is commutative, [14, 2.3]. Moreover, this filtration is compatible with the $\text{ad}U(\mathfrak{g})$ -module structure of $U(\mathfrak{g})$. A generalization of this filtration in the quantum case is the Joseph-Letzter P^+ -filtration, [21]. As in [27],

this will permit us to define the generalized (P^+)-exponents of \mathfrak{g} . First, let's present a few facts on the Joseph-Letzter filtration.

For $\lambda \in P^+$ set

$$F_q^\lambda := \text{ad}U_q(\mathfrak{g}).K_{-2\lambda}, \quad F_{q,\lambda} := \bigoplus_{\nu \leq \lambda} F_q^\nu.$$

Then, $F_{q,\lambda}$ is an $\text{ad}U_q(\mathfrak{g})$ -module. By [18, 7.1.1], $\{F_{q,\lambda}, \lambda \in P^+\}$ defines a P^+ -filtration of $F(U_q(\mathfrak{g}))$. Moreover, let \hat{F}_q^λ , resp. $\hat{F}_q(U_q(\mathfrak{g}))$, be the associated graded space of F_q^λ , resp. $F_q(U_q(\mathfrak{g}))$. As $\text{ad}U_q(\mathfrak{g})$ -modules, \hat{F}_q^λ and F_q^λ are isomorphic. By [20], the Joseph-Letzter's filtration extends to a filtration of $U_q(\mathfrak{g})$ such that $\hat{U}_q(\mathfrak{g})$ is the algebra with the canonical generators, quantum Serre relations, weight relations, and $[\hat{E}_i, \hat{F}_j] = -\delta_{ij} \frac{\hat{K} - \alpha_i}{q - q^{-1}}$. Explicitly this filtration is $\mathcal{F}_\lambda(U_q(\mathfrak{g})) = \bigoplus_{\alpha \geq -2\lambda} U_q(\mathfrak{n}) \otimes K_\alpha \otimes V_q(\mathfrak{n}^-)$ where $V_q(\mathfrak{n}^-)$ is the $\mathbb{C}(q)$ -algebra generated by $K_{\alpha_j} F_j$, $1 \leq j \leq n$. In general, if E is an element, resp. subspace, of $U_q(\mathfrak{g})$, \hat{E} will be the associated graded element, resp. space. Clearly, the algebra $\hat{F}(U_q(\mathfrak{g})) \subset \hat{U}_q(\mathfrak{g})$ admits a Ringel filtration as in 1.3 and the associated graded algebra will be $\text{Gr}\hat{F}_q(U_q(\mathfrak{g}))$.

In $\hat{F}_q(U_q(\mathfrak{g}))$ we have, [*loc. cit.*] :

$$(1.7.1) \quad \hat{F}_q^\lambda \hat{F}_q^{\lambda'} = \hat{F}_q^{\lambda+\lambda'}, \quad \lambda, \lambda' \in P^+.$$

Hence, $\bigoplus_i \hat{F}_q^{\varpi_i}$ generates the graded associated algebra.

1.8. For $\lambda \in P^+$, let z_λ be the unique $\text{ad}U_q(\mathfrak{g})$ -invariant element in $\text{ad}U_q(\mathfrak{g}).K_{-2\lambda}$. By Theorem 1.4, the center Z_q of $U_q(\mathfrak{g})$ is generated (as a space) by the z_λ , $\lambda \in P^+$. Moreover, Z_q is generated as an algebra by the $z_i := z_{\varpi_i}$, [18, 7.1.17].

Definition 1. Let J_q be the ideal of $F(U_q(\mathfrak{g}))$ generated by the $z_i - \varepsilon(z_i)$, $1 \leq i \leq n$. Let $\hat{\mathbb{H}}_q$ be a complementary $\text{ad}U_q(\mathfrak{g})$ -module of \hat{J}_q such that $\hat{\mathbb{H}}_q = \bigoplus_\lambda \hat{\mathbb{H}}_q \cap \hat{F}_q^\lambda$. Let \mathbb{H}_q a corresponding sub- $\text{ad}U_q(\mathfrak{g})$ -module of $F(U_q(\mathfrak{g}))$ such that $\mathbb{H}_q = \bigoplus_\lambda \mathbb{H}_q \cap F_q^\lambda$. For all λ in P^+ , set $\mathbb{H}_q^\lambda = \mathbb{H}_q \cap F_q^\lambda$, and let $\mathbb{H}_q^\lambda(\mu)$, resp. $F_q^\lambda(\mu)$, be the isotypical component of \mathbb{H}_q^λ , resp. F_q^λ , of type $V_q(\mu)$.

Definition 2. Let $\mu \in P^+$. In the algebra $\mathbb{C}[P^+]$ generated by the e^λ , $\lambda \in P^+$, we define the element $Q_\mu(e) = Q_\mu := \sum_\lambda [\mathbb{H}_q^\lambda : V_q(\mu)] e^\lambda$. The dominant weights occurring in this sum with multiplicities will be called P^+ -exponents of μ .

As in the classical case, we have, [18, Lemma 8.1.5] :

$$(1.8.1) \quad [\mathbb{H}_q : V_q(\mu)] = \dim V_q(\mu)_0.$$

Hence, by specializing e on 1, we obtain $Q_\mu(1) = \dim V_q(\mu)_0$. We now give an explicit expression of Q_μ in terms of the canonical base. This is nothing but another formulation of a result of Baumann [1, 3.4]. We give a proof for completion.

Proposition. For μ in P^+ , let $B(\mu)$, resp. $B(\mu)_0$, be the crystal base, resp. the set of 0-weight elements in the crystal base, of $V_q(\mu)$. Then, $Q_\mu = \sum_{b \in B(\mu)_0} e^{\mathcal{E}(b)}$.

Proof. For $\lambda, \mu \in P^+$, consider the following subsets of $B(\mu)_0$.

$$N_{\lambda, \mu} = \{b \in B(\mu)_0, \mathcal{E}(b) \preceq \lambda\}, \quad N_{\lambda, \mu}^0 = \{b \in B(\mu)_0, \mathcal{E}(b) = \lambda\}, \quad N_{\lambda, \mu}^+ = N_{\lambda, \mu} \setminus N_{\lambda, \mu}^0,$$

with cardinals, respectively, $n_{\lambda, \mu}$, $n_{\lambda, \mu}^0$ and $n_{\lambda, \mu}^+$. Now, let's work in the Joseph-Letzter graded algebra. The theorem of separation of variables, [21], gives $\hat{F}_q = \hat{\mathbb{H}}_q \otimes \hat{Z}_q$. Moreover, by semi-simplicity and from the basic properties of the crystal base, [18, 6.3.18] :

$$(1.8.2) \quad [\hat{F}_q^\lambda : V_q(\mu)] = [V_q(\lambda)^* \otimes V_q(\lambda) : V_q(\mu)] = [V_q(\lambda) \otimes V_q(\mu) : V_q(\lambda)] = n_{\lambda, \mu}.$$

We claim that

$$(1.8.3) \quad \hat{F}_q^\lambda(\mu) \simeq n_{\lambda, \mu}^0 V_q(\mu) \oplus \hat{J}_q \cap \hat{F}_q^\lambda(\mu).$$

Indeed, we can prove this fact by induction on the $|\lambda| := \sum_i \lambda_i$, with $\lambda = \sum_i \lambda_i \varpi_i$ in the following way. Suppose this is true for all $\lambda' \in P^+$, $|\lambda'| < |\lambda|$. Then, we have $[\hat{\mathbb{H}}_q(\mu) \cap \hat{F}_q^{\lambda'} : V_q(\mu)] = n_{\lambda', \mu}^0$, for such λ' . Note that \hat{J}_q is the ideal generated by the \hat{z}_{ϖ_i} , [18, 7.3.5]. By the separation of variables theorem and the induction hypothesis, this gives

$$\sum_{\lambda' \prec \lambda} \hat{z}_{\lambda - \mathcal{E}(b)}(\hat{\mathbb{H}}_q \cap \hat{F}_q^{\lambda'}(\mu)) = \hat{J}_q \cap \hat{F}_q^\lambda(\mu).$$

By separation of variables and the induction hypothesis, the left hand term is isomorphic to $\sum_{\lambda' \prec \lambda} n_{\lambda', \mu}^0 V_q(\mu) = n_{\lambda, \mu}^+ V_q(\mu)$. By (1.8.2), this gives (1.8.3) and the proposition is proved. \diamond

Remark. When \mathfrak{g} is of type A_n , then the function Q_μ is the multivariable Kostka-Foulkes polynomial $\mathbb{K}_{\mu, 0}$ defined in [29, 6.2] by Lascoux-Leclerc and Thibon. In fact, the previous proposition materializes their definition in the following sense : \mathbb{K}_μ as defined in [Loc. cit.] is the Hilbert polynomial of the isotypical component of \mathbb{H}_q of type $V_q(\mu)$ in the Joseph-Letzter P^+ -filtration.

1.9. We present a result which can be seen as a quantum version of Dixmier's antihomomorphism [14, 8.4.1].

We consider the Joseph-Letzter associated graded algebra. Let $\hat{\pi}$ be the natural projection from $\hat{F}_q(U_q(\mathfrak{g})) \simeq \oplus_\lambda V_q(\lambda)^* \otimes V_q(\lambda)$ onto $\oplus \text{ad} U_q(\mathfrak{n}) \hat{K}_{-2\lambda} \simeq \oplus V_q(\lambda)^* \otimes v_\lambda$. Then, by [22, Lemma 7.2],

Lemma. *The restriction of $\hat{\pi}$ to $\hat{F}_q(U_q(\mathfrak{g}))^{U_q(\mathfrak{n})}$ is an injective algebra q -antihomomorphism. To be more precise, let $a_\mu \in \hat{F}_q(U_q(\mathfrak{g}))^{U_q(\mathfrak{n})}$, $b_\nu \in (\hat{F}_q^\lambda)^{U_q(\mathfrak{n})}$ of weights, respectively, μ and ν . Then, $\hat{\pi}(a_\mu b_\nu) = q^{(\mu, 2\nu - \lambda)} \hat{\pi}(b_\nu) \hat{\pi}(a_\mu)$. Moreover $\hat{\pi}(\hat{z}_\lambda) = \hat{K}_{-2\lambda}$. \diamond*

For λ in P^+ , let $\hat{\pi}_\lambda$ be the λ component of $\hat{\pi}$. By say [8, Proposition 3.4], it is clear that $\hat{\pi}$ is the restriction of the natural projection on $\hat{U}_q(\mathfrak{n}) \otimes \hat{K}_{-2\lambda}$. Hence, $\hat{\pi}$ is compatible with the Ringel filtration.

2. A-form and specialization.

2.1. The following results can be deduced from [12, 1.5]. Let A be the local algebra $\mathbb{C}[q]_{(q-1)}$. We define the A -form U_A generated by $E_i, F_i, \frac{K_{\varpi_i} - K_{-\varpi_i}}{q - q^{-1}}, K_\lambda$, where $\lambda \in P$, $1 \leq i \leq n$. Remark that U_A is a sub- $\text{ad } U_A$ -module of $U_q(\mathfrak{g})$. Moreover, U_A is A -free and $U_q(\mathfrak{g}) = \mathbb{C}(q) \otimes_A U_A$.

Claim. *Let $\mathbb{C}[P/2P]$ be the algebra of the group $P/2P$, generated as a space by the elements $e^{\bar{\lambda}}, \bar{\lambda} \in P/2P$. There exists a natural isomorphism between $U_A/(q-1)U_A$ and the central extension $\mathbb{C}[P/2P] \otimes U(\mathfrak{g})$ such that the canonical surjection $\phi : U_A \rightarrow U_A/(q-1)U_A \simeq \mathbb{C}[P/2P] \otimes U(\mathfrak{g})$ sends K_λ to $e^{\bar{\lambda}}, \lambda \in P$. \diamond*

Let U_A^+ , resp. U_A^0 , resp. U_A^- , be the positive, resp. Cartan, resp. negative, part of U_A . As in 1.2, the triangular decomposition holds for A -forms.

We define $F(U_A) = U_A \cap F(U_q(\mathfrak{g}))$ and $Z_A := Z \cap U_A$. From the definitions, the algebra Z_A is the center of U_A . Let J_q be the ideal of $F(U_q(\mathfrak{g}))$ generated by the $z_i - \varepsilon(z_i)$, $1 \leq i \leq n$ and set $J_A := J_q \cap F(U_A)$. Then, J_A is a $\text{ad } U_A$ -module. Moreover, the quotient $F(U_A)/J_A \subset F(U_q(\mathfrak{g}))/J$ has no A -torsion.

Proposition. *The morphism ϕ sends*

- (i) $F(U_A)$ onto $U(\mathfrak{g})$, realized as the subalgebra $1 \otimes U(\mathfrak{g})$ in $\mathbb{C}[P/2P] \otimes U(\mathfrak{g})$, with kernel $(q-1)F(U_A)$.
- (ii) J_A onto the minimal primitive ideal $U(\mathfrak{g})\text{Ker}\varepsilon_1$, where ε_1 is the augmentation restricted to $Z(\mathfrak{g})$.

Proof. By Theorem 1.4 (ii), $F(U_A)$ contains $K_{-2\varpi_i}E_{\alpha_i}$, for all i . Hence, by the Claim, $\phi(F(U_A))$ contains $U(\mathfrak{n})$. In the same way, it contains $U(\mathfrak{n}^-)$, and both algebras generate $U(\mathfrak{g})$. Hence, $U(\mathfrak{g}) \subset \phi(F(U_A))$. Now, by [20, Theorem 6.4], $F(U_A) \subset U_A^- \otimes \hat{U}^0 \otimes U_A^+$, with

$$(2.1.1) \quad \hat{U}^0 = \mathbb{C}(q)[K_{2\varpi_i}, K_{-2\varpi_i}] \cap A[K_{\varpi_i}, K_{-\varpi_i}, \frac{K_{\varpi_i} - K_{-\varpi_i}}{q - q^{-1}}] = A[K_{2\varpi_i}, K_{-2\varpi_i}, \frac{1 - K_{-2\varpi_i}}{q - q^{-1}}].$$

Hence, $\phi(F(U_A)) \subset U(\mathfrak{g})$. This gives (i).

Let's prove that $\phi(J_A) = U(\mathfrak{g})\text{Ker}\varepsilon_1$.

$U(\mathfrak{g})\text{Ker}\varepsilon_1 \subset \phi(J_A)$: Let $J'_A = F(U_A)(\text{Ker}\varepsilon \cap Z_A)$. Then, $J_A \subset J'_A$ implies $U(\mathfrak{g})\text{Ker}\varepsilon_1 = \phi(J'_A) \subset \phi(J_A)$.

$\phi(J_A) \subset U(\mathfrak{g})\text{Ker}\varepsilon_1$: Let $M_q(0)$ be the quantum Verma module with highest weight 0, as defined in [18, 3.4.9]. Then, $M_q(0)$ is a cyclic $U_q(\mathfrak{n}^-)$ -free module generated by a highest weight vector v_0 of weight 0. Let $M_A(0)$ be the sub- U_A -module of $M_q(0)$ generated by v_0 . Then, $M_A(0)/(q-1)M_A(0)$ has a natural structure of $U(\mathfrak{g})$ -module by (i). It is easily seen that this is a cyclic $U(\mathfrak{g})$ -module with highest weight 0 and with same formula character than the classical Verma module $M(0)$. Hence, $M_A(0)/(q-1)M_A(0) \simeq M(0)$ as $U(\mathfrak{g})$ -modules. Now, J_A annihilates $M_A(0) \subset M_q(0)$. Hence, $\phi(J_A) \simeq J_A/(q-1)J_A$ annihilates $M(0)$. By Duflo's theorem [14, 8.4.3], we have the desired inclusion. Hence, (ii) holds. \diamond

2.2. Let $U_q(\mathfrak{n})^+$, resp. $U_q(\mathfrak{n}^-)^+$, be the augmentation ideal of $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$. Let φ be the Harish-Chandra map of $U_q(\mathfrak{g})$, i.e. the projection $U_q(\mathfrak{g}) \rightarrow U_q^0$ with kernel $U_q(\mathfrak{g})U_q(\mathfrak{n})^+ + U_q(\mathfrak{n}^-)^+U_q(\mathfrak{g})$. Recall that U_q^0 is isomorphic to $\mathbb{C}(q)[P]$ as a W -module. Recall the so-called twisted action of W on U_q^0 , see [12, 2]. Moreover, by [12, 2.2] :

Proposition. *Let \tilde{U}_A^0 be the intersection of U_A with the $\mathbb{C}(q)$ -algebra generated by the $K_{2\lambda}$, $\lambda \in P$. Then, U_A^0 is a sub- W -module of U^0 and the Harish-Chandra homomorphism φ restricts into an algebra isomorphism $Z_A \simeq (\tilde{U}_A^0)^W$.* \diamond

Remark that the inverse isomorphism can be given explicitly by the Tolstoi projector introduced in [26]. This is implicit in the proof of [12, Proposition 2.2]. The proposition above is the quantum integral analogue of the following classical result, see [14, 7.4] : the Harish-Chandra map establishes an algebra isomorphism between the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ and the W -invariant subalgebra of the Cartan part $U(\mathfrak{h})$ of $U(\mathfrak{g})$.

Theorem. *The A -algebra Z_A is a polynomial algebra over A . It specializes for $q = 1$, onto the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$, realized as the subalgebra $1 \otimes U(\mathfrak{g})$ in $\mathbb{C}[P/2P] \otimes U(\mathfrak{g})$.*

Proof. From Proposition 2.1, Z_A specializes in the subalgebra $1 \otimes U(\mathfrak{g})$. Clearly, specialization sends an element of the center of U_A to an element of $Z(\mathfrak{g})$.

Moreover, \tilde{U}_A^0 specializes onto $U(\mathfrak{h})$ and specialization commutes with W -action. As W is finite, the same is true for the W -invariants. The last assertion of the theorem follows because specialization commutes with the Harish-Chandra homomorphisms.

Let's prove that Z_A is polynomial over A .

Let's identify the center Z of $U(\mathfrak{g})$ with $S(\mathfrak{h})^W$ and the symmetric algebra $S(\mathfrak{h})$ with the algebra of regular functions on the lattice of radical weights Q . In the same way, let's identify Z_q with $(U_q^0)^W$, and U_q^0 with the $\mathbb{C}(q)$ -algebra generated by the functions $K_{\varpi_i}(-) = q^{(\varpi_i, -)}$ on Q . This enables us to give formal expansions at $q = 1$ from elements of Z_q , as in [8, 7.1] and [JL2, 6.13], i.e. we can embed Z_q in $S(\mathfrak{h})^W[[q-1]] \simeq Z[[q-1]]$. Let's consider the augmentation ideal Z^+ of Z . Fix a set of homogeneous elements (C_{m_j}) , $1 \leq j \leq n$, of degree $0 < m_1 \leq \dots \leq m_n$, which is a base of Z^+ modulo $(Z^+)^2$. Let z_i be the quantum central element as in 1.8 and $z'_i = z_i - \varepsilon(z_i)$. From the formula in the proof of [JL2, Lemme 6.15], we have the claim

Claim. *The coefficient of $(q-1)^m$ of the image of z'_i in $Z[[q-1]]$ is an homogeneous element of degree m in Z .* \diamond

Now, the image of z'_i in $Z^+/(Z^+)^2[[q-1]]$ is $\sum a_{ij}C_{m_j}(q-1)^{m_j}$, $a_{ij} \in \mathbb{C}$, $1 \leq i, j \leq n$. Moreover the products $z'_i z'_j$ is zero in this quotient. Recall that the z'_i generate the $\mathbb{C}(q)$ -algebra Z_q and recall that, by the previous arguments, Z_A specializes onto Z at $q = 1$. This implies that there exists a polynomial combination of the z'_i (and of $(q-1)^{-1}$) which specializes on C_{m_j} at $q = 1$. This combination is linear modulo $(Z^+)^2$ and this implies that the matrix (a_{ij}) is invertible. Hence, we can change the index of the z_i in order to obtain non zero minors $(a_{ij})_{1 \leq i, j \leq k}$, $1 \leq k \leq n$. Then, by induction on k , we obtain elements $C_{m_k}^q$ in Z_A which are linear combination of z_k and products of z_i for $i < k$ and which specializes on C_{m_k} . Now, we assert that Z_A is the A polynomial

algebra generated by the $C_{m_k}^q$. Indeed, as in the proof of Lemma A1, this is a direct consequence of the following facts :

- 1) These elements specialize on a polynomial base of Z .
- 2) They generate Z_q as a $\mathbb{C}(q)$ -algebra. \diamond

2.3. Set $F_{A,\lambda} = U_A \cap F_{q,\lambda}$.

Proposition. *We have the following*

- (i) *The A -module $F(U_A)$ is free,*
- (ii) *there exists an $\text{ad}U_A$ -modules gradation $(F_A^\lambda)_{\lambda \in P^+}$ of $F(U_A)$ such that*

$$F(U_A) = \oplus_{\lambda \in P^+} F_A^\lambda, \quad \oplus_{\lambda \leq \mu} F_A^\lambda = U_A \cap F_{q,\mu}.$$

Proof. The module U_A is A -free. In particular, $F_{A,\lambda}$ has no A -torsion, so it is a free A -module. Fix $\lambda \leq \mu$ in P^+ . Then, $F_{A,\mu}/F_{A,\lambda}$ embeds in $F_{q,\mu}/F_{q,\lambda}$, so it has no A -torsion. Hence, the A -module $F_{A,\lambda}$ is a direct summand in $F_{A,\mu}$. Remark that $F(U_A) = \bigcup_{\mu} F_{A,\mu}$ to obtain (i) and (ii). \diamond

Remark. $F(U_A)$ strictly contains $\oplus_{\lambda}(U_A \cap F_q^\lambda)$. For example, set $\mathfrak{g} = \mathfrak{sl}_2$ and let $z_\varpi \in F_q^\varpi$ be the Casimir element corresponding to the quantum trace of the fundamental module $V_q(\varpi)$. Then, [8, 5.3], $\phi(z_\varpi) = 2$, $\frac{z_\varpi - 2}{q - q^{-1}}$ belongs to $F(U_A)$ but not to $\oplus_{\lambda} U_A \cap F_q^\lambda$.

2.4. We prove in this section that the separation theorem at the quantum level, i.e. on $\mathbb{C}(q)$, and on the classical level implies the separation theorem for the A -forms. The proof relies on general properties of torsion-free modules over A -polynomial algebras. These properties are given in the Appendix.

Theorem. *The separation theorem holds for the A -forms, i.e. via multiplication*

$$Z_A \otimes H_A \simeq F(U_A),$$

for an $\text{ad}U_A$ -module H_A . Moreover, H_A can be obtained as $U_A \cap \mathbb{H}_q$, where \mathbb{H}_q is as in [1.8, Definition 1].

Proof. Fix \mathbb{H}_q as in Definition 1 and set $H_A = U_A \cap \mathbb{H}_q$. It is sufficient to prove the theorem on the isotypical component, i.e $F_A(\lambda) = Z_A \otimes H_A(\lambda)$, $\lambda \in P^+$. Now, from Theorem 2.2, the center Z_A is a polynomial ring over A . We are in the framework of [Appendix, Proposition A4] with $T = q - 1$ and $R = Z_A$. From the quantum and the classical theorem of separation of variables, [21] and [27], the hypothesis of Proposition A4 are verified with $r = \dim V(\lambda) \dim V(\lambda)_0$. Hence, $F_A(\lambda)$ is a Z_A -free module with rank r .

Fix a base of this module. The theorem is a consequence of the following facts :

- (i) $H_A(\lambda)$ generates the Z_q module $F_q(\lambda)$.
- (ii) $H_A(\lambda)$ is a A -direct summand in $F_A(\lambda)$.

Indeed, fix a A -base of $H_A(\lambda)$ which is compatible with the gradation of $F_A(\lambda)$. Then, this base generates the Z_q module $F_q(\lambda)$ by [18, 7.3.8]. This implies (i). Moreover, by construction $H_A(\lambda)$ is a free A -module with finite rank and

$F(U_A)(\lambda)/H_A(\lambda)$ has no $(q-1)$ -torsion. So, (ii) holds.

Now, fix a A -base of $H_A(\lambda)$ and let d be the determinant of the matrix of this base in the fixed base of the Z_A -module $F_A(\lambda)$. Then, (i) asserts that $(q-1)^m d \in A^*$ for some positive integer m . By (ii), we obtain $m = 0$. This ends the proof. \diamond

Let ϕ be the morphism of specialization as in 2.1. We have :

Corollary. *The morphism ϕ sends*

- (i) H_A onto a complementary subspace of $U(\mathfrak{g})\text{Ker}\varepsilon_1$ in $U(\mathfrak{g})$,
- (ii) The μ isotypical component of H_A onto the μ isotypical component of some module of harmonic elements of $U(\mathfrak{g})$ (in the sense of 0.4).

Proof. Remark that $Z_A = (Z_A \cap \text{Ker}\varepsilon) \oplus A$ as A -modules. By the previous theorem, this implies that $F(U_A) = J_A \oplus H_A$. Hence, Proposition 2.1 gives the corollary. \diamond

2.5. From Theorem 2.4, we can construct a graduation $(F_A^\lambda)_{\lambda \in P^+}$ such that $H_A = \bigoplus H_A \cap F_A^\lambda$ and $Z_A = \bigoplus Z_A \cap F_A^\lambda$. Let Z_A^λ and H_A^λ be the graded component of Z_A and H_A . We have

$$(2.5.1) \quad \bigoplus_{\mu' \preceq \mu \leq \lambda} Z_A^{\mu-\mu'} \otimes H_A^{\mu'} = F_{A,\lambda}.$$

For all $\lambda \in P^+$, set $F_\lambda = F_\lambda(U(\mathfrak{g})) = \phi(F_{A,\lambda}) \in U(\mathfrak{g})$. This is an $\text{ad}U(\mathfrak{g})$ -submodule of $U(\mathfrak{g})$. In general, we omit the index A to note the image by ϕ , i.e. we note $E := \phi(E_A)$. We want to prove that F_λ is a filtration of $U(\mathfrak{g})$ which is analogue to the Joseph-Letzter's filtration in the quantum case. Indeed we have :

Proposition. *The algebra $U(\mathfrak{g})$ is filtered by the $\text{ad}U(\mathfrak{g})$ -modules $(F_\lambda)_{\lambda \in P^+}$. Moreover,*

- (i) $F_\lambda \simeq \bigoplus_{\mu \leq \lambda} F^\mu$
- (ii) $F_\mu \simeq V(\mu)^* \otimes V(\mu)$
- (iii) $\bigoplus_{\mu' \preceq \mu \leq \lambda} Z^{\mu-\mu'} \otimes H^{\mu'} = F_\lambda$.

Proof. The $\text{ad}U_A$ modules $(F_{A,\lambda})_{\lambda \in P^+}$ give a filtration of $F(U_A)$. Hence, by specialization and by Proposition 2.3, we have the first assertion. (i) and (ii) are obtained by specialization, using the fact that the A -spaces $F_{A,\lambda}$ and F_A^λ are A -summands in $F(U_A)$.

(iii) is obtained from (2.5.1). \diamond

3. Gröbner bases and harmonic elements.

3.1. Recall the definition of roots packages, see 1.5. In the graded algebra $\text{Gr}U_q(\mathfrak{g})$, the generators $\text{Gr}z_i$ of $\text{Gr}Z_q$ are given by :

Proposition. *For $1 \leq i \leq n$, we have, up to a multiplicative scalar*

$$\text{Gr}z_i = \left(\prod_{\beta \in \Delta_i^+} \text{Gr}E_\beta \text{Gr}F_\beta \right) \text{Gr}K_{-2\varpi_i}.$$

The $\text{Gr}z_i$, $1 \leq i \leq n$, generate $\text{Gr}Z_q$.

Proof. Recall that z_i corresponds to the quantum trace on $V_q(\varpi_i)$, [8, 5.1]. The formula is a direct consequence of [8, Proposition 3.4], and [11, Lemma 4.1 (iii)]. From 1.5, $\{\Delta_i^+, i\}$ is a partition of Δ^+ , so the $\text{Gr} \prod_i z_i^{a_i}$, $a_i \in \mathbb{N}$ are linearly independant. Hence, they generate $\text{Gr } Z_q$ as a space. This proves the proposition. \diamond

3.2.

Definition. Let \mathcal{M} be the set of monomials M in $U_q(\mathfrak{g})$ such that

- (i) $M = \prod_{j,m} E_{\alpha_{j,m}}^{a_{j,m}} K_{\lambda} F_{\alpha_{j,m}}^{b_{j,m}}$, $0 \leq a_{j,m}, b_{j,m}$, $\lambda \in P$,
- (ii) For all j , $\prod_m a_{j,m} b_{j,m} = 0$.

Then, the following separation theorem holds.

Theorem. Let J_q^0 be the ideal of $U_q(\mathfrak{g})$ generated by the z_i , $1 \leq i \leq n$. Fix a reduced decomposition of w_0 . Then :

- (i) The set $\{z_i, 1 \leq i \leq n\}$ is a Gröbner base of J_q^0 for the lexicographic ordering of the PBW-base, i.e. $u \in J_q^0 = \langle z_i \rangle \Rightarrow \text{Gr } u \in \langle \text{Gr } z_i \rangle$.
- (ii) The algebra $\text{Gr } U_q(\mathfrak{g})$ is free over $\text{Gr } Z_q$ with base $\text{Gr } \mathcal{M}$.
- (iii) The algebra $U_q(\mathfrak{g})$ is free over Z_q with base \mathcal{M} .

Proof. First remark that, by construction and by Proposition 3.1, the image of $\text{Gr } \mathcal{M}$ in $\text{Gr } U_q(\mathfrak{g}) / \langle \text{Gr } z_i, 1 \leq i \leq n \rangle$ is a base of this quotient.

Let $\text{Gr } z_j$, resp. $\text{Gr } m_j$, be monomials in $\text{Gr } Z_q$, resp. $\text{Gr } \mathcal{M}$, $j = 1, 2$. Then, $\{\Delta_i^+, i\}$ being a partition of Δ^+ , we obtain by Proposition 3.1 :

$$(3.2.1) \quad \text{Gr } z_1 \text{Gr } m_1 = \text{Gr } z_2 \text{Gr } m_2 \Rightarrow \text{Gr } z_1 = \text{Gr } z_2 \text{ and } \text{Gr } m_1 = \text{Gr } m_2.$$

Now, considering $\text{Gr } U_q(\mathfrak{g})$ as a $\mathbb{N}^N \times P \times \mathbb{N}^N$ -graded algebra, see 1.3, we remark that each graded component of $\text{Gr } U_q(\mathfrak{g})$ has dimension 0 or 1. Hence, (3.2.1) implies that $\text{Gr } U_q(\mathfrak{g})$ is free over $\text{Gr } Z_q$, with base $\text{Gr } \mathcal{M}$. This implies (ii) and then (iii).

Let Z_q^+ be the ideal of Z_q generated by the z_i . Then, by (iii), each element u of J_q^0 can be decomposed in a unique way into $u = \sum_k P_k m_k$, with $P_k \in Z_q^+$ and $m_k \in \mathcal{M}$. This implies (i) by (3.2.1). \diamond

Remark 1. It is possible to generalize the theory of Gröbner bases in some non-commutative contexts, [23]. (i) can also be proved in this context by using the first Buchsberger criterion [2]. Indeed, Proposition 3.1 asserts that the initial monomials of the z_i are relatively prime.

Remark 2. Unfortunately, the specialization of the center is not compatible with the associated graded algebra. Indeed, the initial terms of the generators z_i may vanish at $q = 1$. The base given in Theorem 3.2 (iii) has no analogue in the classical case.

3.3. This section will be of no use in the sequel. The results of 1.3 generalize when $q = \varepsilon$ is a root of one. We define the quantum algebra $U_\varepsilon(\mathfrak{g})$ as in [12, 1.5]. By the description of the center Z_ε of quantum groups at roots of one, see [13], we obtain :

Theorem. When $q = \varepsilon$ is a root of one, the algebra $\text{Gr } U_\varepsilon(\mathfrak{g})$ is free over the graded algebra $\text{Gr } Z_\varepsilon$. This implies that $U_\varepsilon(\mathfrak{g})$ is free over its center. \diamond

For example, when ε is a l th root of one, with l odd, then a base of $U_\varepsilon(\mathfrak{g})$ over Z_ε is given by $\mathcal{M}^l := \{M \in \mathcal{M}, M = \prod_{i,j,m} E_{\alpha_{j,m}}^{a_{j,m}} K_{\varpi_i}^{c_i} F_{\alpha_{j,m}}^{b_{j,m}}, 0 \leq a_{j,m}, b_{j,m}, c_i < l\}$.

3.4. Let π as in 1.9. The next proposition gives a base of $\pi(\mathbb{H}_q^{U_q(\mathfrak{n})})$ in terms of the dual of the canonical base of $U_q(\mathfrak{n}^-)$, (see 1.6). When u is a weight vector in a $U_q(\mathfrak{g})$ -module, then we note $\Omega(u)$ the weight of u .

Proposition. *For λ in P^+ , set*

$$\mathcal{B}_H := \{b, b \in \mathcal{B}, \mathcal{E}(b) \preceq -\Omega(b)\}, \quad \mathcal{B}_H^\lambda := \{b, b \in \mathcal{B}_H, \mathcal{E}(\bar{b}) = \lambda\}.$$

Then, a base of $\pi(\mathbb{H}_q^{U_q(\mathfrak{n})})$ is given by $(K_{-2\lambda} b_\lambda^, \lambda \in P^+, b_\lambda \in \mathcal{B}_H^\lambda)$ for some harmonic space \mathbb{H}_q .*

Proof. Fix $\mu = \sum \mu_i \varpi_i \in P^+$. Recall the notations of Proposition 1.8. we have

$$(3.4.1) \quad \dim(\mathbb{H}_q^\lambda(\mu))^{U_q(\mathfrak{n})} = [\mathbb{H}_q^\lambda(\mu) : V_q(\mu)] = n_{\lambda,\mu}^0.$$

Via the natural embedding of $B(\mu)$ in the canonical base, we have by [24, Proposition 8.2] $N_{\lambda,\mu}^0 = \{b, b \in \mathcal{B}, \Omega(b) = -\mu, \mathcal{E}(\bar{b}) \preceq \mu, \mathcal{E}(b) = \lambda\}$. In the same way, $\dim(\hat{F}_q^\lambda)^{U_q(\mathfrak{n})}(\mu) = n_{\lambda,\mu}$ and $N_{\lambda,\mu} = \{b, b \in \mathcal{B}, \Omega(b) = -\mu, \mathcal{E}(\bar{b}) \preceq \mu, \mathcal{E}(b) \preceq \lambda\}$.

Claim. *A base of $\pi(F_q^\lambda(\mu))^{U_q(\mathfrak{n})}$ is given by $(K_{-2\lambda} b^*)$, where b runs over the set $\overline{N}_{\lambda,\mu}$, i.e. the image of $N_{\lambda,\mu}$ by the involution $\bar{}$ described in 1.6.*

Proof of the claim. Fix λ, μ, ξ in P^+ , $\lambda = \sum \lambda_i \varpi_i, \mu = \sum \mu_i \varpi_i$. By say, [18, 6.3.20] the image of the highest weight space of weight μ in $V_q(\xi) \otimes V_q(\lambda)$ by the natural projection on $V_q(\xi) \otimes v_\lambda$ is $(V_q(\xi))_{\mu-\lambda}^\lambda \otimes v_\lambda$, where

$$(V_q(\xi))_{\mu-\lambda}^\lambda := \{v \in V_q(\xi), \Omega(v) = \mu - \lambda, F_i^{\mu_i+1} v = 0, \forall i\}$$

$$(3.4.2) \quad = \{v \in V_q(\xi), \Omega(v) = \mu - \lambda, E_i^{\lambda_i+1} v = 0, \forall i\}.$$

We have the succession of space isomorphisms $(V_q(\lambda)^*)_{\mu-\lambda}^\lambda = \langle b^* \in \mathcal{B}(\lambda)^*, \Omega(b) = -\mu, \mathcal{E}(b) \preceq \mu \rangle = \langle b^* \in \mathcal{B}^*, \Omega(b) = -\mu, \mathcal{E}(\bar{b}) \preceq \lambda, \mathcal{E}(b) \preceq \mu \rangle = \overline{N}_{\lambda,\mu}$. Indeed, the first equality comes from [30, Theorem 4.4 (c), 4.3] and (3.4.2), the second one is deduced from [24, Proposition 8.2]. The claim results from Theorem 1.4 (ii) and Theorem 1.6. \diamond

end of proof of the proposition. Suppose ν in P^+ , $\nu \preceq \lambda$. In the graded algebra $\hat{F}_q(U_q(\mathfrak{g}))$, $\langle K_{-2\lambda} b^*, b^* \in \mathcal{B}(\nu)_\mu^* \rangle \subset \hat{\pi}(\hat{J}_q)$ by the claim and Lemma 1.9. Then, the proposition results from the claim by (3.4.1). \diamond

Remark 1. What is behind the proof is a natural twist $\tau : B(\lambda) \otimes B(\mu) \rightarrow B(\mu) \otimes B(\lambda)$ defined to be the unique crystal map such that the image of the \tilde{e}_i -invariant elements, $1 \leq i \leq n$, is given by $\tau(v_\lambda \otimes b v_\mu) = v_\mu \otimes \bar{b} v_\lambda$, for $\mathcal{E}(b) \preceq \lambda, \mathcal{E}(\bar{b}) \preceq \mu$. It would be interesting to understand the braiding of this twist on $B(\varpi)^{\otimes k}$, where ϖ is a basic weight.

Remark 2. We can prove that the generators of $\mathbb{H}_q^{U_q(\mathfrak{n})}$ can be calculated from the $K_{-2\lambda} b_\lambda^*$, $\lambda \in P^+$, $b_\lambda \in \mathcal{B}_H^\lambda$, by applying the Tolstoi projector, see [26].

3.5. We suppose in this section that \mathfrak{g} is of type A-D-E and that the decomposition of w_0 corresponds to an orientation of the Coxeter graph of \mathfrak{g} . In this case, from Ringel's Hall algebra approach, [33], the canonical base can be labelled by the Kostant partitions. We present a few results which can be found in [31], [32, par 2 and 3]. Recall that $\{E_\psi, \psi \in \mathbb{N}^N\}$ is a base of $U_q(\mathfrak{n})$. There exists a parametrization $\mathcal{B} = \{b_\psi, \psi \in \mathbb{N}^N\}$ of the canonical base such that :

$$(3.5.1) \quad \text{Gr } b_\psi^* = \text{Gr } E_\psi, \quad \text{up to a multiplicative scalar in } \mathbb{Z}[q],$$

$$(3.5.2) \quad \text{Gr } b_\psi^* b_\phi^* = \text{Gr } q^{[\psi, \phi]} b_{\psi+\phi}^*, \quad \text{for a power } [\psi, \phi] \text{ of } q,$$

$$(3.5.3) \quad \mathcal{E}(b_{\psi+\phi}) \preceq \mathcal{E}(b_\psi) + \mathcal{E}(b_\phi).$$

The following definition flows from Proposition 3.4 :

Definition. The harmonic cone is defined by $\mathcal{C}_0 := \{\psi, b_\psi \in \mathcal{B}_H\} \subset \mathbb{N}^N$.

For ψ in \mathbb{N}^N , set $\Omega(\psi) = \sum_{l=1}^N \psi_l \beta_l$ be the weight of E^ψ and $\mathcal{E}(\psi) = \mathcal{E}(\bar{b}_\psi)$. Let $<, >$ be the bilinear form defined by the $\Delta^+ \times \Delta^+$ matrix $M = (m_{\alpha, \beta})$, with

$$m_{\alpha, \beta} = \begin{cases} 0 & \text{if } \beta \leq \alpha, \\ (\alpha, \beta) & \text{if not} \end{cases}$$

Theorem. *The algebra $\mathbb{C}[\mathcal{C}_0]$ of the semigroup \mathcal{C}_0 is normal. Via the anti-homomorphism $\text{Gr } \hat{\pi}$, see 1.9, the algebra $\text{Gr}(\hat{F}_q(U_q(\mathfrak{g}))/\hat{J})^{U_q(\mathfrak{n})}$ is isomorphic to the following algebra : a base is $a_\psi, \psi \in \mathcal{C}_0$ with the multiplication given by*

$$a_\psi a_\phi = \begin{cases} 0 & \text{if } \mathcal{E}(\psi + \phi) \prec \mathcal{E}(\psi) + \mathcal{E}(\phi), \\ q^{<\psi, \phi> - 2(\Omega(\psi), \mathcal{E}(\phi))} a_{\psi+\phi} & \text{if } \mathcal{E}(\psi + \phi) = \mathcal{E}(\psi) + \mathcal{E}(\phi). \end{cases}$$

Proof. The cone \mathcal{C}_0 is $\{\psi \in \mathbb{N}^N, \mathcal{E}(b_\psi) \preceq \Omega(\psi)\}$. Ω is clearly additive on \mathbb{N}^N and so, \mathcal{C}_0 is a semigroup by (3.5.3). For all ψ in \mathbb{N}^N , set $a_\psi := \text{Gr } \hat{\pi}(\text{Gr } \hat{K}_{-2\mathcal{E}(\psi)} \hat{E}^\psi)$. The set $\{a_\psi, \psi \in \mathcal{C}_0\}$ is a base of $\text{Gr } \hat{\pi}(\text{Gr}(\hat{F}_q(U_q(\mathfrak{g}))/\hat{J})^{U_q(\mathfrak{n})})$ by Proposition 3.4 and the previous formulas. The multiplication rule is given by Lemma 1.9 and (1.3.2).

By [5, Theorem 3.8], we obtain that $\mathcal{E}(k\psi) = k\mathcal{E}(\psi)$, $\psi \in \mathbb{N}^N$. . This gives $k\psi \in \mathcal{C}_0$, $k \in \mathbb{N}^*$, $\psi \in \mathbb{N}^N \Rightarrow \psi \in \mathcal{C}_0$. Hence, $\mathbb{C}[\mathcal{C}_0]$ is normal. \diamond

4. The A_n case.

4.1. Let \mathfrak{g} of type A_n . Set $F_q = F_q^{\varpi_1}$. For all $\lambda = \sum \lambda_i \varpi_i \in P$, we define $h(\lambda) = \sum_i i \lambda_i$. Define the total order \leq on P^+ such that

- 1) $h(\lambda) < h(\mu) \Rightarrow \lambda \leq \mu$ and
 2) Fix $\lambda = \sum_i \lambda_i \varpi_i$, $\mu = \sum_i \mu_i \varpi_i$, with $h(\lambda) = h(\mu)$. Then, $\lambda \leq \mu$ iff $(\lambda_1, \dots, \lambda_n)$ is lower than (μ_1, \dots, μ_n) for the reverse lexicographical order of \mathbb{N}^n .

Example. For $\mathfrak{g} = \mathfrak{sl}_4$, $0 \leq \varpi_1 \leq \varpi_2 \leq 2\varpi_1 \leq \varpi_3 \leq \varpi_1 + \varpi_2 \leq 3\varpi_1 \leq \varpi_1 + \varpi_3 \leq 2\varpi_2 \leq 4\varpi_1 \leq \dots$

It is easily seen that this ordering verifies the hypothesis of 1.1.

We have :

Lemma. For all k in \mathbb{N} , $\sum_{0 \leq m \leq k} F_q^m = \bigoplus_{h(\lambda) \leq k} F_q^\lambda$.

Proof. It is well known that the minimal integer k such that $V_q(\lambda) \subset V_q(\varpi_1)^{\otimes k}$ is $h(\lambda)$. Hence the lemma results from [19, Corollary 3.10]. \diamond

4.2. This section is devoted to the specialization of the center of $U_q(\mathfrak{sl}_{n+1})$. Recall that the center Z_q of $U_q(\mathfrak{sl}_{n+1})$ is generated as a space by the z_λ , $\lambda \in P^+$, defined as in 3.1 and as an algebra by the $z_i \in F_q^{\varpi_i}$, $1 \leq i \leq n$. Recall the following theorem of [8, Proposition 6.3, Thorne 7.3].

Theorem. There exists an $n \times n$ matrix \mathcal{A} with coefficients in A and a column matrix $C_q = (C_{q,i})_{1 \leq i \leq n}$ with $C_{q,i} \in Z_A$ such that

- (i) $\mathcal{A}C_q = (z_i - C_{n+1}^i)_{1 \leq i \leq n}$,
- (ii) $\phi(C_{q,i})$, $1 \leq i \leq n$, has degree $i + 1$ in Z and they generate the polynomial algebra Z ,
- (iii) $\mathcal{A} = (a_{i,j}(q - q^{-1})^{j-1})$, with $a_{i,j} = 1^j C_{n+1}^{i-1} - 2^j C_{n+1}^{i-2} + \dots + (-1)^{i-1} i^j C_{n+1}^0$ \diamond

4.3. In the following, we set $U := U(\mathfrak{sl}_{n+1})$. The aim of this section is to describe the P^+ -filtration $(F_\lambda)_{\lambda \in P^+}$ of $U(\mathfrak{sl}_{n+1})$ in terms of the natural \mathbb{N} -filtration $(U_k)_{k \in \mathbb{N}}$ of $U(\mathfrak{sl}_{n+1})$.

For $\lambda = \sum_i \lambda_i \varpi_i \in P^+$, set $z(\lambda) = \sum (i + 1) \lambda_i$. Set $J := \phi(J_A)$. Recall that, by Proposition 2.1 (ii), J is a minimal primitive ideal. By Lemma 4.1, we can choose \mathbb{H}_q in Theorem 2.4 such that

$$\mathbb{H}_q \cap \bigoplus_{h(\lambda) \leq k} F_q^\lambda \subset \sum_{0 \leq m \leq k} (\mathbb{H}_q^{\varpi_1})^m,$$

for all k . Set $H_A = \mathbb{H}_q \cap U_A$, as in Theorem 2.4. Set H_A^λ and Z_A^λ as in 2.5. By construction, $H_A \cap (\sum_{h(\lambda) \leq k} F_{q,\lambda}) = \bigoplus_{h(\lambda) \leq k} H_A^\lambda$ and $Z_A \cap (\sum_{z(\lambda) \leq k} F_{q,\lambda}) = \bigoplus_{z(\lambda) \leq k} Z_A^\lambda$.

Proposition. Set $U = U(\mathfrak{sl}_{n+1})$, $H = \phi(H_A)$, $H_k = H \cap U_k$, $Z_k = Z \cap U_k$, $k \in \mathbb{N}$, we have

- (i) $Z \otimes H = U$ via multiplication,
- (ii) $\phi(\sum_{h(\lambda) \leq k} H_A^\lambda) = H_k$,
- (iii) $\phi(\sum_{z(\lambda) \leq k} Z_A^\lambda) = Z_k$,

Proof. (i) follows from Proposition 2.5 (iii), Corollary 2.4 and Theorem 2.2.

Let's prove (ii). For k in \mathbb{N} , set $F_{A,k} = F(U_A) \cap (\sum_{h(\lambda) \leq k} F_{q,\lambda})$. Let $\bar{\phi}$ be the quotient morphism $\bar{\phi} : F(U_A)/J_A \rightarrow U/J$. Then, $\bar{\phi}$ is surjective. We want to show that $\bar{\phi}(\overline{F_{A,k}})$

is the image \overline{U}_k of U_k by the canonical surjection. First of all, $\overline{F_q^{\varpi_1}}$ is simple and contains \overline{E}_1 . Hence, $\overline{\phi}(\overline{F_{A,\varpi_1}}) = \mathbb{C} \oplus \mathfrak{sl}_{n+1}$ and $\overline{\phi}(\bigoplus_{m \leq k} \overline{(F_{A,\varpi_1})^m}) = \overline{U}_k$. By Lemma 4.1, this gives

$$(4.3.1) \quad \overline{U}_k \subset \overline{\phi}(\overline{F_{A,k}}).$$

For μ in P^+ and m in \mathbb{N} , let c_μ^m be the coefficient of q^m in the Kostka-Foulkes function $K_{\mu,0}(q)$. For $\lambda \in P^+$, let $c_\mu^\lambda = n_{\lambda,\mu}^0$, be the coefficient of e^λ in Q_μ . Then, by the Hesselink formula and the Lascoux-Leclerc-Thibon theorem, see 0.3 :

$$\dim \overline{U}_k = \sum_{\mu \in P^+} \sum_{m \leq k} c_\mu^m \dim V(\mu) = \sum_{\mu \in P^+} \sum_{m \leq k} \sum_{h(\lambda)=m} c_\mu^\lambda \dim V_q(\mu).$$

So, by 1.8, Proposition 2.3 and Theorem 2.4, $\dim \overline{U}_k = \dim \overline{F_{q,k}} = \dim(\phi(\overline{F_{A,k}}))$. Hence, we have equality in (4.3.1). This gives (ii) by the hypothesis on \mathbb{H}_q .

Let's prove (iii). Fix s , $1 \leq s \leq n$. Let Δ_s be the determinant of the matrix $(a_{i,j})_{1 \leq i,j \leq s}$, see 4.2. It is an easy exercise to use the Van der Monde determinant to obtain

$$\Delta_s = (-1)^{\frac{s(s-1)}{2}} \prod_{m=1}^s m! \neq 0.$$

From Theorem 4.2, we obtain that $C_{q,s} = D_{q,s}$ modulo $(q-1)$, for all $1 \leq s \leq n$, with $D_{q,s} \in (q - q^{-1})^{-(s-1)} \sum_{i=1}^s \mathbf{Q}(z_i - C_{n+1}^i)$. The elements $\phi(D_{q,s}) = \phi(C_{q,s})$, $1 \leq s \leq n$, generate the algebra $Z(\mathfrak{g})$. So, by induction on the P^+ -filtration of Z_A , $D_{q,s}$, $1 \leq s \leq n$, generate the algebra Z_A by using Nakayama's lemma on the finite rank A -modules $Z_A \cap F_{q,m}$, $m \in \mathbb{N}$. The left hand term in (ii) is generated by $\prod_s D_{q,s}^{n_s}$, with $\sum (s+1)n_s \leq k$. This implies (ii). \diamond

Set $H^\lambda = \phi(H_A^\lambda)$, resp. $Z^\lambda = \phi(Z_A^\lambda)$. This gives the following theorem :

Theorem. *The \mathbb{N} -natural filtration and the P^+ -filtration of U , see 2.4, can be compared as follow :*

$$U_k = \bigoplus_{h(\lambda)+z(\mu) \leq k} H^\lambda \otimes Z^\mu.$$

\diamond

4.4. In this section, we illustrate, in the case \mathfrak{sl}_{n+1} , some of the main results and techniques of this article.

Examples 1-2-3 are devoted to the cone \mathcal{C}_0 .

See [25] for the bijection between the canonical base and semi-standard Young tableaux and for the calculation of $\mathcal{E}(b)$, where b is associated to a given tableau. See [4, Theorem 4.1] for inequations in Example 3.

Example 1. For $\mathfrak{g} = \mathfrak{sl}_2$, the cone \mathcal{C}_0 is \mathbb{N} , where $n \in \mathbb{N}$ corresponds in the crystal base to the semi-standard tableau of shape $2n\varpi$ and of weight 0.

$$\boxed{1} \boxed{\dots} \boxed{\dots} \boxed{1} \boxed{2} \boxed{\dots} \boxed{\dots} \boxed{2}$$

Example 2. For $\mathfrak{g} = \mathfrak{sl}_3$, and $w_0 = s_1 s_2 s_1$, the cone \mathcal{C}_0 is generated as a semi-group by $(1, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(2, 0, 1)$, corresponding to the following tableaux of weight 0.

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array}$$

In the $\mathbb{C}(q)$ algebra of \mathcal{C}_0 , the multiplication corresponds to concatenation. We have the relation :

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} * \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array}$$

In the degenerated algebra :

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array} = 0$$

This gives, for the case \mathfrak{sl}_3 , an interpretation of a result of Joseph-Letzter, [22].

Example 3. For $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $w_0 = s_1 \dots s_n s_1 \dots s_{n-1} \dots s_1 s_2 s_1$. Let $\alpha_{i,j} := \alpha_i + \dots \alpha_j$, $1 \leq i \leq j \leq n$, be the set of positive roots. Then \mathcal{C}_0 is the set of points $(n_{i,j}, 1 \leq i \leq j \leq n)$ such that

$$\sum_{k=j}^n n_{i,k} - \sum_{k=j+1}^n n_{i+1,k} \leq \left(\sum_{k \leq k'} n_{k,k'} \alpha_{k,k'}, \alpha_i \right), \quad 1 \leq i \leq j \leq n.$$

In Example 4, we calculate P^+ -exponents of type μ , $\mu \in P^+$, in a particular case. Then, we deduce the \mathbb{N} -exponents of μ .

Example 4. For \mathfrak{g} of type A_3 , let's calculate the \mathbb{N} -exponents of $\mu = \varpi_1 + 2\varpi_2 + \varpi_3$. By [loc. cit., 3.4.2 (i)], $B(\mu)_0$ is parametrized by the semi-standard Young tableaux of shape μ with entries $(1, 1, 2, 2, 3, 3, 4, 4)$. These tableaux are given by

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 4 & \\ \hline 4 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & \\ \hline 4 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & 4 & \\ \hline 3 & & & \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & \\ \hline 4 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 4 & \\ \hline 3 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 4 & & & \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 4 & \\ \hline 3 & & & \\ \hline \end{array}$$

By [loc. cit., 3.4.2 (ii)] and Proposition 1.8, the P^+ -exponents of μ are respectively $2\varpi_1 + \varpi_3$, $\varpi_1 + \varpi_2$, $\varpi_1 + \varpi_2 + \varpi_3$, $\varpi_1 + \varpi_2 + \varpi_3$, $\varpi_1 + 2\varpi_3$, $2\varpi_2$, $\varpi_2 + \varpi_3$. After reordering, the exponents of μ are $(3, 4, 5, 5, 6, 6, 7)$.

Example 5 is devoted to the decription of the Joseph-Letzter filtration for \mathfrak{sl}_2 .

Example 5. For $\mathfrak{g} = \mathfrak{sl}_2$, we have $F_\varpi = \mathbb{C} \oplus \mathfrak{g} \oplus \mathbb{C}c$, where c is the quadratic Casimir element. $F_{n\varpi} = F_\varpi^n$. $F_{n\varpi}$ can be described by its \mathfrak{n}^+ -invariant component $\bigoplus_{k \leq n} (F^{k\varpi})^{\mathfrak{n}^+} = \bigoplus_{0 \leq i \leq k \leq n} \mathbb{C}E^{k-i}c^i$. Moreover, $U_n = \bigoplus_{0 \leq h+z \leq n} H^{h\varpi} \otimes Z^{2z\varpi}$.

APPENDIX

Lemma A1. Let M be a module on a noetherien local domain R , \mathfrak{m} be its maximal ideal, K be its fraction field and k be its residue field. Suppose $\dim_K K \otimes_R M = \dim_k k \otimes_R M = r$. Then, M is a free R -module with rank r .

Proof. Let (b_i) , $1 \leq i \leq r$, be a part of M such that $(1 \otimes b_i)$ is a base of the k -space $k \otimes_R M$. Then, (b) is A -free. By cardinality, it is a base of the K -space $K \otimes_R M$. Let N be the A -module generated by (b) . Then, M/N is \mathfrak{m} -torsion.

Let's prove that M/N has no non zero \mathfrak{m} -torsion elements. Let m in M be the lift of a \mathfrak{m} -torsion element in M/N . For all m in M , there exists r such that $\mathfrak{m}^r m \in N$. Choose r minimal non zero. By tensoring by k , and using the fact that each element of $R - \mathfrak{m}$ is invertible, we obtain that $(1 \otimes b)$ is not free, which is absurd. Hence, $r = 0$ and this gives the Lemma. \diamond

Lemma A2. Let K be a field and let R be a localization of the K -algebra $K[T]$ by a multiplicative part S such that $T \notin S$. Let M be a R -module. Suppose that

- 1) $M[T^{-1}]$ is $R[T^{-1}]$ -free with rank r ,
- 2) M/TM is a r -dimensional R/TR -space. Then, M is a free R -module with rank r .

Proof. First of all, we prove the following assertion.

Assertion. There exist e_1, \dots, e_r in M such that $N = \bigoplus R e_i$ is free with rank r , $\{k \otimes e_i\}$ is a base of M/TM and $N[T^{-1}] = M[T^{-1}]$.

We proceed by induction. Let e_1^0, \dots, e_r^0 in M such that $(k \otimes e_i^0)$ is a base of the k -space $k \otimes_R M$. Set $N_0 = \bigoplus R e_i^0 \subset M$. If $N_0[T^{-1}] = M[T^{-1}]$, there is nothing to prove. If not, then there exists x_1 in M such that $T^k x_1 \notin N_0$ for all k in \mathbb{Z} . Set $N_1 = N_0 + R x_1$. Then, N_1 is a finitely generated torsion free R -module, so, it is a free R -module. Localizing by T , we obtain that $\text{rk} N_1 = r$. There exists a base e_1^1, \dots, e_r^1 of the R module N_1 and, as N_1 contains N_0 , $(k \otimes e_i^1)$ is a base of the k -space $k \otimes_R M$. Using induction and the fact that $M[T^{-1}]$ is a noetherian $R[T^{-1}]$ -module gives the assertion.

We want to prove that the module N of the assertion is equal to M . It is sufficient to prove that $N_{\mathfrak{m}} = M_{\mathfrak{m}}$ for all maximal ideal \mathfrak{m} of R . If $T \notin \mathfrak{m}$, the equality results from $N[T^{-1}] = M[T^{-1}]$. If $T \in \mathfrak{m}$, then $M_{\mathfrak{m}}/N_{\mathfrak{m}}$ is T -torsion by the previous formula.

Moreover, as in the proof of the previous lemma, $M_{\mathfrak{m}}/N_{\mathfrak{m}}$ has no nonzero T -torsion elements. This ends the proof. \diamond

Proposition A4. Define the polynomial ring $R_1 = \mathbb{C}[X_1, \dots, X_n]$ and the local ring $R_2 = \mathbb{C}[T]_{(T)}$. Let $K_1 = \mathbb{C}(X_1, \dots, X_n)$ et $K_2 = \mathbb{C}(T)$ be their fraction fields. Set $R = R_1 \otimes R_2$ endowed with its natural structure of ring. Let M torsion-free R -module such that

- (i) M/TM , is a free R/TR -module with rank r ,
- (ii) $K_2M = M_T$ is a free $R_1 \otimes K_2$ -module with rank r .

Then, M is a free R -module with rank r .

Proof. From Lemma A2, we deduce that K_1M is a free K_1R -module with rank r . We now have to prove that M is finitely generated over R . This is the aim of the next lemma :

Lemma A3. Let M torsion-free R -module such that K_1M , resp. $K_2M = M_T$, is a free $K_1 \otimes R_2$ -module with rank r , resp. $R_1 \otimes K_2$ -module with rank r . Then, M is finitely generated as a R -module.

Proof. First remark that R is a UFD domain. Recall that for all multiplicative part S of R , we have $S^{-1}(\bigwedge^r M) = \bigwedge^r(S^{-1}M)$, as $S^{-1}M$ -modules. Let $F = \bigwedge^r M$. From the hypothesis, we have that K_1F and K_2F are free with rank 1. Let $e_1^s, \dots, e_r^s \in M$ such that e_i^s is a base of the K_sR -module K_sM , $s = 1$ or 2 . Set $e^s := e_1^s \wedge \dots \wedge e_r^s \in F$. The element e^s is a base of the K_sR module K_sF . Let x be in F . We have $x = (\alpha_s/a_s)e^s$ and $e^2 = (\gamma/d)e^1$ with $\alpha, \gamma \in R$, $a_s \in R_s$, $d \in R_1$, $\text{GCD}(\alpha_s, a_s) = \text{GCD}(\gamma, d) = 1$. Hence, we obtain $\alpha_1/a_1 = (\alpha_2\gamma)/(bd)$ and then, $r := \alpha_1d/a_1 = \alpha_2\gamma/b \in K_1R \cap K_2R = R$. In conclusion, F is a subset of $(1/d)e^1$.

Fix an element y in M , $y = \sum_i (\mu_i/p_i)e_i^1$, $\mu_i \in R$, $p_i \in R_1$, $\text{GCD}(\mu_i, p_i) = 1$. By the previous assertion, there exists s_i such that

$$e_1^1 \wedge \dots \wedge e_{i-1}^1 \wedge y \wedge e_{i+1}^1 \wedge \dots \wedge e_r^1 = (\mu_i/p_i)e^1 = (s_i/d)e^1.$$

We deduce that $\mu_i/p_i = s_i/d$ and so $y \in (1/d) \sum Re_i$. We obtain that M is finitely generated as a R -module. \diamond

end of proof of proposition. In order to show the proposition, it suffices, by the Quillen-Suslin theorem, cf. [28, Introduction], to prove that M is a projective, i.e. locally free, R -module. Let \mathfrak{m} be a maximal ideal of R . Let's prove that $M_{\mathfrak{m}}$ is free. If T does not belong to \mathfrak{m} , then $M_{\mathfrak{m}}$ is a localization of the module $M[T^{-1}] = K_2M$ and so, it is a free module. If T belongs to \mathfrak{m} , it is sufficient to prove that $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$ -module. This results from the hypothesis and [16, Corollary 6.9]. \diamond

ACKNOWLEDGMENTS

I thank R. Berger and M. Cretin for helpful discussions on Gröbner bases and M. Reineke for e-mail conversations on Hall algebras. I am grateful to B. Leclerc and S.

Zelikson for several informations about crystal bases. I also wish to thank J. Germoni for comments and encouragements. Most of all, I am in debt to P. Baumann for his remarks and corrections. And last, but not least, the article owes a lot to the careful remarks of the referee.

BIBLIOGRAPHY

1. P. BAUMANN. *Another proof of Joseph and Letzter's separation of variables theorem for quantum groups*, preprint.
2. T. BECKER and V. WEISPFENNING. *Gröbner bases : a computational approach to commutative algebra*, Graduate texts in Mathematics, 141, New-York, Springer-Verlag, (1993).
3. D.J. BENSON. *Representations and cohomology I, Basic representation theory of finite groups and associative algebras*, Cambridge studies in Advanced Mathematics, 30, Cambridge University Press.
4. A. BERENSTEIN and A. ZELEVINSKY. *Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics.*, Duke Math. J., 82, (1996), no. 3, 473-502.
5. A. BERENSTEIN and A. ZELEVINSKY. *Tensor product multiplicities, Canonical bases and Totally positive varieties*, ArXiv:math.RT/9912012.
6. N. BOURBAKI. *Groupes et Algèbres de Lie, Chap. VI*, Masson, Paris, (1981).
7. R.K. BRYLINSKI. *Limits of weight spaces, Lusztig's q -analogs, and fiberings of adjoint orbits*, J. Amer. Math. Soc., 2, (1989), 517-533.
8. P. CALDERO. *Générateurs du centre de $U_q(sl(N + 1))$* , Bull. Sci. Math., 118, (1994), 177-208.
9. P. CALDERO. *Elements ad-finis de certains groupes quantiques*, C. R. Acad. Sci. Paris, t. 316, Serie I, (1993), 327-329.
10. P. CALDERO. *Etude des q -commutations dans l'algèbre $U_q(\mathfrak{n})$* , J. Algebra, 178, (1995), 444-457.
11. P. CALDERO. *On the q -commutations in $U_q(\mathfrak{n})$ at roots of one*, J. Algebra, Vol 210, (1998), 557-576.
12. C. DE CONCINI and V. G. KAC. *Representations of quantum groups at roots of 1*, Colloque Dixmier, Progress in Math., 92, (1990), 471-506.
13. C. DE CONCINI, V. G. KAC, C. PROCESI. *Quantum coadjoint action*, J. Amer. Math. Soc., 5, (1992), 151-189.
14. J. DIXMIER. *Algèbres enveloppantes*, Cahiers scientifiques n° 37, Gauthiers-Villars, Paris, 1974.
15. V. G. DRINFELD. *On almost cocommutative Hopf algebras*, Leningrad Math. J., Vol. I, (1990), n° 2, 321-342.

16. D. EISENBUD. *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, 150, Berlin: Springer-Verlag. xvi, (1995).
 17. W.H. HESSELINK. *Characters of the Nullcone*, Math. Ann., 252, (1980), 179-182.
 18. A. JOSEPH. *Quantum groups and their primitive ideals*, Springer-Verlag, 29, (1995).
 19. A. JOSEPH. *On the Mock Peter-Weyl theorem and the Drinfeld double of a double*, Reine Angew. Math., 507, (1999), 37-56.
 20. A. JOSEPH and G. LETZTER. *Local finiteness of the adjoint action for quantized enveloping algebras*, J. Algebra, 153, (1992), 289-318.
 21. A. JOSEPH and G. LETZTER. *Separation of variables for quantized enveloping algebras*, Amer. J. Math., 116, (1994), 127-177.
 22. A. JOSEPH and G. LETZTER. *Verma modules annihilators for quantized enveloping algebras*, Ann. Sci. Ec. Norm. Sup., 28, N^o 4, (1995), 493-526.
 23. A. KANDRI-RODY and V. WEISPFENNING. *Non-commutative Gröbner bases in Algebras of solvable type*, J. Symb. Comp., 9, (1990), 1-26.
 24. M. KASHIWARA. *On Crystal Bases*, Canad. Math. Soc., Conference Proceed., 16, (1995), 155-195.
 25. M. KASHIWARA and T. NAKASHIMA. *Crystal graphs for representations of the q -analogue of classical Lie algebras*, J. Alg., 165, N^o 2, (1994), 295-345.
 26. S.M. KHOROSHKIN and V.N. TOLSTOI. *The uniqueness theorem for the Universal \mathcal{R} -matrix*, Lett. Math. Phys., 24, n^o 3, (1992), 231-244.
 27. B. KOSTANT. *Lie groups representations on polynomial rings*, Amer. J. Math., 85, (1963), 327-404.
 28. T.Y. LAM. *Serre's conjecture*, Lecture Notes in Mathematics, 635, Berlin - Heidelberg - New York: Springer-Verlag. XV, (1978).
 29. A. LASCOUX, B. LECLERC and J-Y. THIBON. *Crystal graphs and q -analogues of Weight Multiplicities for the Root System A_n* , Lett. Math. Phys., 35, (1995), 359-374.
 30. G. LUSZTIG. *Canonical bases arising from quantized enveloping algebras. II*, Prog. Theor. Phys. Suppl., 102, (1990), 175-201.
 31. G. LUSZTIG. *Introduction to quantum groups*, Progress in Mathematics, 110, Birkhäuser, (1993).
 32. M. REINEKE. *Multiplicative Properties of Dual Canonical Bases of Quantum Groups*, J. Alg., 211, (1999), 134-149.
 33. C.M. RINGEL. *Hall algebras and quantum groups*, Invent. Math., 101, (1990), 583-592.
 34. M. ROSSO. *Analogues de la forme de Killing et du théorème de Harish-Chandra pour les groupes quantiques*, Ann. Sci. Ec. Norm. Sup., 23, (1990), 445-467.
- Institut Girard Desargues, UPRES-A-5028*
Université Claude Bernard Lyon I, Bat 101
69622 Villeurbanne Cedex, France

e-mail : caldero@desargues.univ-lyon1.fr

Keywords : Quantum groups, enveloping algebras, canonical base, harmonic space.